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On the Addition Theorems of Jacobi and Weierstrass.

BY E. STUDY.

In the paper "Sphärische Trigonometrie, orthogonale Substitutionen und elliptische Functionen" (Leipzig, Hirzel, 1893) an investigation has been made of the addition theorems of elliptic θ -functions due to Jacobi and Weierstrass, in order to simplify the statement of these theorems, considered as a whole, and to bring them into connection with certain geometrical ideas. The importance of the subject, which forms, as is well known, the true basis of a great and most beautiful part of the theory of elliptic functions, may justify our attempt to give still a new presentation of the matter.

To Mr. Caspari we owe the important remark that the said addition theorems may be derived from simple algebraic identities by means of a quadratic transformation (Math. Annalen, Bd. 28, S. 495). We shall make a liberal use of this principle, deducing from it not single formulæ, but at once the complete system of addition theorems, as stated in the above paper, without referring, however, to the theory of transformation.

Since the author's "Trigonometry" cannot be supposed to be in the hands of the readers of this journal, we may repeat the main definitions. The notation introduced here is, as far as the same ground is covered, exactly the same that has been used by Tannery and Molk in their "Traité des fonctions elliptiques" (I, Paris, 1893), to which the reader may be referred. Harkness and Morley, in their "Treatise on the Theory of Functions" (New York and London, 1893), have introduced a very similar system of notation, but a certain difference is caused by their supposing $\Re\left(\frac{\omega_2}{\omega_1}\right) > 0$, whereas we suppose $\Re\left(\frac{\omega_3}{\omega_1}\right) > 0$.

In two subsequent papers I purpose to extend these considerations to some more complicated addition theorems in the theory of elliptic and hyper-

elliptic functions, where new and important results can be derived from the principle of applying a transformation of the period 3 instead of the usual involutory transformation of the arguments.

Denoting by ω_1 and ω_3 the primitive periods ω and ω' of Weierstrass, and by ω_2 the negative sum of these periods $\omega_2 = -\omega'' = -\omega - \omega'$, we have

$$\omega_1 + \omega_2 + \omega_3 = 0.$$

Now let λ, μ, ν denote the indices 1, 2, 3 in any one of the three arrangements (1, 2, 3); (2, 3, 1); (3, 1, 2): we have

$$\begin{aligned} \sqrt[4]{e_\mu - e_\nu} &= \frac{\sigma_\nu \omega_\mu}{\sigma \omega_\mu} = -\frac{e^{-\eta_\nu \omega_\mu} \sigma \omega_\lambda}{\sigma \omega_\mu \sigma \omega_\nu}, & \sqrt[4]{e_\nu - e_\mu} &= \frac{\sigma_\mu \omega_\nu}{\sigma \omega_\nu} = -\frac{e^{-\eta_\mu \omega_\nu} \sigma \omega_\lambda}{\sigma \omega_\mu \sigma \omega_\nu}, \\ \therefore \sqrt[4]{e_\nu - e_\mu} &= i \sqrt[4]{e_\mu - e_\nu}. \end{aligned} \quad (1)$$

Denote further the quantity $-e^{-\frac{\pi i}{4}}$ by $\sqrt{-i}$, and let the signs $\sqrt[4]{e_\mu - e_\nu}$ be chosen in accordance with (1) and with the equations

$$\sigma \omega_\lambda = \frac{\sqrt{-i} e^{\frac{1}{2} \eta_\lambda \omega_\lambda}}{\sqrt[4]{e_\nu - e_\lambda} \sqrt[4]{e_\lambda - e_\mu}} \quad (2)$$

derived from (1): write finally

$$\sqrt[4]{e_\nu - e_\mu} = \sqrt{i} \sqrt[4]{e_\mu - e_\nu}, \quad (3)$$

where \sqrt{i} means the one or the other of the two values of this radical. Then the equations

$$\begin{aligned} \Theta u &= \sqrt[4]{e_\mu - e_\nu} \sqrt[4]{e_\nu - e_\lambda} \sqrt[4]{e_\lambda - e_\mu} \cdot \sigma u = \sqrt[8]{G} \cdot \sigma u, \\ \Theta_\lambda u &= \sqrt[4]{e_\mu - e_\nu} \cdot \sigma_\lambda u, \quad \Theta_\mu u = \sqrt[4]{e_\nu - e_\lambda} \cdot \sigma_\mu u, \quad \Theta_\nu u = \sqrt[4]{e_\lambda - e_\mu} \cdot \sigma_\nu u, \end{aligned} \quad (4)$$

in which λ, μ, ν now may denote 1, 2, 3 in an arbitrary but fixed order, define a set of four functions, $\Theta, \Theta_1, \Theta_2, \Theta_3$, which we shall use instead of Weierstrass' functions $\sigma, \sigma_1, \sigma_2, \sigma_3$.

The Θ -functions (4) are interchanged and at the same time multiplied by

certain factors, if we augment the argument u by a half period; the formulæ in question are

$$\begin{aligned}\Theta(u \pm \omega_\lambda) &= \pm (\sqrt{-i})^\epsilon \cdot e^{\frac{1}{2}\eta_\lambda \omega_\lambda} e^{\pm \eta_\lambda u} \cdot \Theta_\lambda u, \\ \Theta_\lambda(u \pm \omega_\lambda) &= \mp (\sqrt{-i})^{-\epsilon} \cdot e^{\frac{1}{2}\eta_\lambda \omega_\lambda} e^{\pm \eta_\lambda u} \cdot \Theta u, \\ \Theta_\mu(u \pm \omega_\lambda) &= (\sqrt{-i})^\epsilon \cdot e^{\frac{1}{2}\eta_\lambda \omega_\lambda} e^{\pm \eta_\lambda u} \cdot \Theta_\nu u, \\ \Theta_\nu(u \pm \omega_\lambda) &= (\sqrt{-i})^{-\epsilon} \cdot e^{\frac{1}{2}\eta_\lambda \omega_\lambda} e^{\pm \eta_\lambda u} \cdot \Theta_\mu u,\end{aligned}\tag{5}$$

etc., with *cyclical* permutation of λ, μ, ν .

The exponent ϵ has the value $+1$ or -1 , namely

$$\epsilon = \frac{(e_\mu - e_\nu)(e_\nu - e_\lambda)(e_\lambda - e_\mu)}{(e_2 - e_3)(e_3 - e_1)(e_1 - e_2)}.\tag{6}$$

The Θ functions are one-valued functions of u , but two-valued functions if also the ratio $\omega:\omega'$ is considered as variable (like the functions Θ of Mr. Weierstrass). But we may easily transform them into one-valued functions of all their arguments by multiplying them with a properly chosen function of $\omega:\omega'$. Taking, for instance, $G^{\frac{1}{2}} = \sqrt[4]{e_\mu - e_\nu} \sqrt[4]{e_\nu - e_\lambda} \sqrt[4]{e_\lambda - e_\mu}$ as a multiplier, we obtain one-valued homogeneous functions of the degree -2 satisfying the equation $\Theta'(0)^2 = \Theta_1(0)\Theta_2(0)\Theta_3(0)$ instead of the equation $\Theta'(0) = \Theta_1(0)\Theta_2(0)\Theta_3(0)$, satisfied by the Θ -functions as defined above; and taking $G^{-\frac{1}{2}}$ as a multiplier, we obtain one-valued homogeneous functions of the degree zero, satisfying the equation $1 = \Theta_1(0)\Theta_2(0)\Theta_3(0)$. The advantage of the above system of notation is simply this, that it enables us to express the quantity $\sqrt[4]{e_\mu - e_\nu}$ by $\Theta_\lambda(0)$.

Our functions $\Theta, \Theta_1, \Theta_2, \Theta_3$ are very similar to the functions $\Theta_1, \Theta_2, \Theta_3, \Theta_0$ used in the collection of formulæ edited by Mr. Schwarz, but they differ from them in some respects. They have especially this property, not to be found in the functions employed by Weierstrass, that from every formula which is right for a given arrangement of the marks λ, μ, ν , a right formula may be deduced by a *cyclical* permutation of the marks. It follows from this that we must decompose some of our sets of formulæ into three different sets if we prefer to apply the original notation of Mr. Weierstrass.

We start from a set of four equations, which are really a special case of the formulæ to be demonstrated, but can easily be derived from the very principles of this theory,* since they contain only two arguments, u, v :

* See, for instance, Weber, *Theorie der elliptischen Functionen*, II, §19. (Braunschweig, 1893.)

$$\begin{aligned}
& \Theta_\mu(0) \Theta_\nu(0) [\Theta(2u) \Theta_\lambda(2v) + \Theta_\lambda(2u) \Theta(2v)] \\
&= 2\Theta(u+v) \Theta_\lambda(u+v) \Theta_\mu(u-v) \Theta_\nu(u-v), \\
& \Theta_\mu(0) \Theta_\nu(0) [\Theta(2u) \Theta_\lambda(2v) - \Theta_\lambda(2u) \Theta(2v)] \\
&= 2\Theta_\mu(u+v) \Theta_\nu(u+v) \Theta(u-v) \Theta_\lambda(u-v), \\
& \Theta_\mu(0) \Theta_\nu(0) [\Theta_\mu(2u) \Theta_\nu(2v) + \Theta_\nu(2u) \Theta_\mu(2v)] \\
&= 2\Theta_\mu(u+v) \Theta_\nu(u+v) \Theta_\mu(u-v) \Theta_\nu(u-v), \\
& \Theta_\mu(0) \Theta_\nu(0) [\Theta_\mu(2u) \Theta_\nu(2v) - \Theta_\nu(2u) \Theta_\mu(2v)] \\
&= -2\Theta(u+v) \Theta_\lambda(u+v) \Theta(u-v) \Theta_\lambda(u-v).
\end{aligned} \tag{7}$$

Introducing here the abbreviation

$$T_{0\lambda}u = \Theta u \cdot \Theta_\lambda u, \quad T_{\mu\nu}u = \Theta_\mu u \cdot \Theta_\nu u,$$

we may write the equations (7) in the following form :

$$\begin{aligned}
\Theta_\mu(0) \Theta_\nu(0) \cdot \Theta(u+v) \Theta_\lambda(u-v) &= T_{0\lambda}u T_{\mu\nu}v + T_{\mu\nu}u T_{0\lambda}v, \\
\Theta_\mu(0) \Theta_\nu(0) \cdot \Theta_\lambda(u+v) \Theta(u-v) &= T_{0\lambda}u T_{\mu\nu}v - T_{\mu\nu}u T_{0\lambda}v, \\
\Theta_\mu(0) \Theta_\nu(0) \cdot \Theta_\mu(u+v) \Theta_\nu(u-v) &= T_{\mu\nu}u T_{\mu\nu}v - T_{0\lambda}u T_{0\lambda}v, \\
\Theta_\mu(0) \Theta_\nu(0) \cdot \Theta_\nu(u+v) \Theta_\mu(u-v) &= T_{\mu\nu}u T_{\mu\nu}v + T_{0\lambda}u T_{0\lambda}v.
\end{aligned} \tag{8}$$

Hence the four Θ -products on the left are connected with certain two functions ϕu and ψu by equations of the following peculiar form :

$$\begin{aligned}
\Theta(u+v) \Theta_\lambda(u-v) &= \mathfrak{U} \cdot \mathfrak{B} \cdot (U + V) \\
\Theta_\lambda(u+v) \Theta(u-v) &= \mathfrak{U} \cdot \mathfrak{B} \cdot (-U + V), \\
\Theta_\mu(u+v) \Theta_\nu(u-v) &= \mathfrak{U} \cdot \mathfrak{B} \cdot (-1 + UV), \\
\Theta_\nu(u+v) \Theta_\mu(u-v) &= \mathfrak{U} \cdot \mathfrak{B} \cdot (1 + UV),
\end{aligned} \tag{9}$$

where, for sake of simplicity, U, V stand for $\phi(u), \phi(v)$ and $\mathfrak{U}, \mathfrak{B}$ for $\psi(u), \psi(v)$.

Augmenting u and v by $\frac{\omega_\lambda}{2}$, we find that a similar system of four equations holds for the products $\Theta_\kappa(u+v) \Theta_\kappa(u-v)$:

$$\begin{aligned}
\Theta(u+v) \Theta(u-v) &= \mathfrak{U} \cdot \mathfrak{B} \cdot (U - V), \\
\Theta_\lambda(u+v) \Theta_\lambda(u-v) &= \varepsilon i \cdot \mathfrak{U} \cdot \mathfrak{B} \cdot (U + V), \\
\Theta_\mu(u+v) \Theta_\mu(u-v) &= \mathfrak{U} \cdot \mathfrak{B} \cdot (1 + UV), \\
\Theta_\nu(u+v) \Theta_\nu(u-v) &= -\varepsilon i \cdot \mathfrak{U} \cdot \mathfrak{B} \cdot (1 - UV).
\end{aligned} \tag{10}$$

u , U and \mathfrak{U} , V have here, of course, not the same meaning they have in the formulæ (9).

Now, between such equations as (9) or (10), written with different arguments, we are able to eliminate the quantities u , U , etc., by means of the following elementary theorem:

The quantities

$$\begin{aligned} r_0 &= (U - U_1)(U_2 - U_3), \\ r_1 &= -(U + U_1)(U_2 + U_3), \\ r_2 &= (1 + UU_1)(1 + U_2U_3), \\ r_3 &= -(1 - UU_1)(1 - U_2U_3), \end{aligned} \quad (11, I)$$

and the corresponding quantities \mathfrak{r}_κ , \mathfrak{u}_κ , derived from r_κ by cyclical permutation of the indices 1, 2, 3, are connected by the following linear equations:

$$\begin{aligned} 0 = \\ \begin{array}{cccc} r_0 + \mathfrak{u}_0 + \mathfrak{u}_0, & r_1 + \mathfrak{u}_2 + \mathfrak{u}_3, & r_2 + \mathfrak{u}_3 + \mathfrak{u}_1, & r_3 + \mathfrak{u}_1 + \mathfrak{u}_2, \\ r_1 + \mathfrak{u}_3 + \mathfrak{u}_2, & r_0 - \mathfrak{u}_1 + \mathfrak{u}_1, & r_3 + \mathfrak{u}_0 - \mathfrak{u}_3, & -r_2 + \mathfrak{u}_2 + \mathfrak{u}_0, \\ r_2 + \mathfrak{u}_1 + \mathfrak{u}_3, & -r_3 + \mathfrak{u}_3 + \mathfrak{u}_0, & r_0 - \mathfrak{u}_2 + \mathfrak{u}_2, & r_1 + \mathfrak{u}_0 - \mathfrak{u}_1, \\ r_3 + \mathfrak{u}_2 + \mathfrak{u}_1, & r_2 + \mathfrak{u}_0 - \mathfrak{u}_2, & -r_1 + \mathfrak{u}_1 + \mathfrak{u}_0, & r_0 - \mathfrak{u}_3 + \mathfrak{u}_3, \end{array} \end{aligned} \quad (12)$$

or, in a different arrangement,

$$\begin{aligned} r_0 + r_\alpha &= -\mathfrak{u}_0 + \mathfrak{u}_\alpha = -\mathfrak{u}_\beta - \mathfrak{u}_\gamma, \\ r_0 - r_\alpha &= \mathfrak{u}_\beta + \mathfrak{u}_\gamma = -\mathfrak{u}_0 - \mathfrak{u}_\alpha, \\ r_\beta + r_\gamma &= -\mathfrak{u}_0 - \mathfrak{u}_\alpha = \mathfrak{u}_0 - \mathfrak{u}_\alpha, \\ r_\beta - r_\gamma &= \mathfrak{u}_\beta - \mathfrak{u}_\gamma = \mathfrak{u}_\beta - \mathfrak{u}_\gamma, \end{aligned} \quad (\alpha, \beta, \gamma = 1, 2, 3) \quad (13)$$

or finally,

$$\begin{aligned} -\mathfrak{u}_0 - \mathfrak{u}_1 - \mathfrak{u}_2 - \mathfrak{u}_3 &= 2r_0 = -\mathfrak{u}_0 + \mathfrak{u}_1 + \mathfrak{u}_2 + \mathfrak{u}_3, \\ \mathfrak{u}_0 + \mathfrak{u}_1 - \mathfrak{u}_2 - \mathfrak{u}_3 &= 2r_1 = -\mathfrak{u}_0 + \mathfrak{u}_1 - \mathfrak{u}_2 - \mathfrak{u}_3, \\ \mathfrak{u}_0 - \mathfrak{u}_1 + \mathfrak{u}_2 - \mathfrak{u}_3 &= 2r_2 = -\mathfrak{u}_0 - \mathfrak{u}_1 + \mathfrak{u}_2 - \mathfrak{u}_3, \\ \mathfrak{u}_0 - \mathfrak{u}_1 - \mathfrak{u}_2 + \mathfrak{u}_3 &= 2r_3 = -\mathfrak{u}_0 - \mathfrak{u}_1 - \mathfrak{u}_2 + \mathfrak{u}_3, \end{aligned} \quad (14)$$

etc. (with cyclical permutation of r , \mathfrak{u} , \mathfrak{u}).

Before applying this to the Θ -functions we may, for sake of convenience, still change slightly the expressions r_κ . Writing $-U$ or $\frac{1}{U}$ instead of U , and

neglecting factors common to all the quantities x, y, z , we obtain two other sets of 12 quantities enjoying the same property :

(11, II).

$$\begin{aligned} r_0 &= -(U + U_1)(U_2 - U_3), & r_0 &= (1 - UU_1)(U_2 - U_3), \\ r_1 &= (U - U_1)(U_2 + U_3), & r_1 &= -(1 + UU_1)(U_2 + U_3), \\ r_2 &= (1 - UU_1)(1 + U_2U_3), & r_2 &= (U + U_1)(1 + U_2U_3), \\ r_3 &= -(1 + UU_1)(1 - U_2U_3), & r_3 &= -(U - U_1)(1 - U_2U_3). \end{aligned}$$

(11, III).

Now, we substitute in the equations (11, I) and (11, II) instead of the quantities $U_\kappa = \phi(u_\kappa)$ their expressions in terms of the functions Θ from (10), and in the same way we introduce into (11, III) the Θ -products taken from (9). Then, suppressing again factors common to all the quantities x, y, z , we obtain the following expressions of 12 quantities connected by the equations (12), (13), (14):

(15, I).

$$\begin{aligned} r_0 &= \Theta(u_2 - u_3) \Theta(u_2 + u_3) \Theta(u + u_1) \Theta(u - u_1) = \Theta a \Theta b \Theta c \Theta d, \\ r_1 &= \Theta_\lambda(u_2 - u_3) \Theta_\lambda(u_2 + u_3) \Theta_\lambda(u + u_1) \Theta_\lambda(u - u_1) = \Theta_\lambda a \Theta_\lambda b \Theta_\lambda c \Theta_\lambda d, \\ r_2 &= \Theta_\mu(u_2 - u_3) \Theta_\mu(u_2 + u_3) \Theta_\mu(u + u_1) \Theta_\mu(u - u_1) = \Theta_\mu a \Theta_\mu b \Theta_\mu c \Theta_\mu d, \\ r_3 &= \Theta_\nu(u_2 - u_3) \Theta_\nu(u_2 + u_3) \Theta_\nu(u + u_1) \Theta_\nu(u - u_1) = \Theta_\nu a \Theta_\nu b \Theta_\nu c \Theta_\nu d, \end{aligned}$$

(15, II).

$$\begin{aligned} r_0 &= -\Theta(u_2 - u_3) \Theta(u_2 + u_3) \Theta_\lambda(u + u_1) \Theta_\lambda(u - u_1) = \Theta a \Theta b \Theta_\lambda c \Theta_\lambda d, \\ r_1 &= \Theta_\lambda(u_2 - u_3) \Theta_\lambda(u_2 + u_3) \Theta(u + u_1) \Theta(u - u_1) = -\Theta_\lambda a \Theta_\lambda b \Theta c \Theta d, \\ r_2 &= -\Theta_\mu(u_2 - u_3) \Theta_\mu(u_2 + u_3) \Theta_\nu(u + u_1) \Theta_\nu(u - u_1) = -\Theta_\mu a \Theta_\mu b \Theta_\nu c \Theta_\nu d, \\ r_3 &= \Theta_\nu(u_2 - u_3) \Theta_\nu(u_2 + u_3) \Theta_\mu(u + u_1) \Theta_\mu(u - u_1) = \Theta_\nu a \Theta_\nu b \Theta_\mu c \Theta_\mu d. \end{aligned}$$

(15, III).

$$\begin{aligned} r_0 &= \Theta(u_2 - u_3) \Theta_\lambda(u_2 + u_3) \Theta_\mu(u + u_1) \Theta_\nu(u - u_1) = \Theta a \Theta_\lambda b \Theta_\mu c \Theta_\nu d, \\ r_1 &= -\Theta_\lambda(u_2 - u_3) \Theta(u_2 + u_3) \Theta_\nu(u + u_1) \Theta_\mu(u - u_1) = \Theta_\lambda a \Theta b \Theta_\nu c \Theta_\mu d, \\ r_2 &= \Theta_\mu(u_2 - u_3) \Theta_\nu(u_2 + u_3) \Theta(u + u_1) \Theta_\lambda(u - u_1) = \Theta_\mu a \Theta_\nu b \Theta c \Theta_\lambda d, \\ r_3 &= -\Theta_\nu(u_2 - u_3) \Theta_\mu(u_2 + u_3) \Theta_\lambda(u + u_1) \Theta(u - u_1) = \Theta_\nu a \Theta_\mu b \Theta_\lambda c \Theta d. \end{aligned}$$

The quantities y_κ, z_κ are, in each case, derived from the quantities x_κ by cyclical permutation of u_1, u_2, u_3 . The quantities a, b, c, d introduced here, and the

corresponding quantities a', b', c', d' and a'', b'', c'', d'' which enter into the expressions of the quantities \mathfrak{y} and \mathfrak{z} , are defined as follows:

$$\begin{aligned} a &= u_2 - u_3, & b &= -u_2 - u_3, & c &= u + u_1, & d &= -u + u_1, \\ a' &= u_3 - u_1, & b' &= -u_3 - u_1, & c' &= u + u_2, & d' &= -u + u_2, \\ a'' &= u_1 - u_2, & b'' &= -u_1 - u_2, & c'' &= u + u_3, & d'' &= -u + u_3, \end{aligned} \quad (16)$$

These quantities, forming the arguments of the Θ -functions $x, \mathfrak{y}, \mathfrak{z}$ as defined by (15, I) or (15, II) or (15, III), are connected by the same linear equations that are satisfied by the functions $x, \mathfrak{y}, \mathfrak{z}$, namely,

$$\begin{aligned} -a'' - b'' - c'' - d'' &= 2a = -a' + b' + c' + d', \\ a'' + b'' - c'' - d'' &= 2b = -a' + b' - c' - d', \\ a'' - b'' + c'' - d'' &= 2c = -a' - b' + c' - d', \\ a'' - b'' - c'' + d'' &= 2d = -a' - b' - c' + d', \text{ etc.} \end{aligned} \quad (17)$$

This is the new formulation of the theorems of Jacobi and Weierstrass, as given in the author's Trigonometry. Its peculiarity consists in the statement of the whole system of 256 equations of the type of Weierstrass' equations with three terms, and in the bringing of these formulæ into close connection with the formulæ of Jacobi's; finally, in the division of all these formulæ into $1 + 9 + 6 = 16$ families. We indicate briefly, for the sake of completeness, the main features of this arrangement.

The expressions (15, I) define one single family. The formulæ (15, II) correspond to 9 families, derived from one another by cyclical permutations of λ, μ, ν and b, c, d . Finally (8, III) represents 6 families, since λ, μ, ν in *these* formulæ (as well as in (15, I)) may be taken in any order whatever.

Our formulæ are interchanged among each other if we augment the quantities a, b, c, d by multiples of $\omega_\lambda, \omega_\mu, \omega_\nu$, the sum of which is a multiple of a whole period. These substitutions constitute, of course, a *group* of commutative operations containing an infinite number of substitutions. Augmenting, however, one of the arguments by a double period $4\tilde{\omega}$, or two of the arguments by the same simple period $2\tilde{\omega}$, we transform every one of our formulæ into itself. Considering these substitutions, which form evidently a *subgroup* of the above group, as not different from the identical operation, we have in our group only a finite number, namely 256 substitutions. This finite group now contains a subgroup of 16 substitutions which interchange with one another only such formulæ as

belong to the same family. The generating operations of this subgroup consist in augmenting one argument by a simple period $2\tilde{\omega}$, and in augmenting all arguments a, b, c, d by the same half period $\tilde{\omega}$. Considering finally also these operations as not differing from identity, our group is reduced to 16 substitutions which interchange the 16 families, each family looked upon as a whole.

For further details and applications we refer to the last chapter of the paper mentioned above.

The process applied here to the demonstration of Jacobi's and Weierstrass's theorems is perhaps the most rapid one, producing all these formulæ at one time, and allowing a full insight into the structure of the whole system. But we must not forget that the above algebraic theorem, although utterly simple, is by no means obvious; it has apparently not hitherto presented itself. As a matter of fact, the Θ -functions are in general easier to deal with than such algebraic expressions.

JOHNS HOPKINS UNIVERSITY, *February, 1894.*